

FINITE SUMS OF COMMUTATORS

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ABSTRACT. We show that elements of unital C^* -algebras without tracial states are finite sums of commutators. Moreover, the number of commutators involved is bounded, depending only on the given C^* -algebra.

1. INTRODUCTION

It was shown in [2] that in finite von Neumann algebras, elements with central trace zero are sums of at most 10 commutators. The C^* -algebra case was considered in [1]. The main result there states that if the unit of a C^* -algebra A is properly infinite (i.e. there exists two orthogonal projections $p, q \in A$ equivalent to 1), then any hermitian element is a sum of at most five self-adjoint commutators. In this paper we consider the more general case of unital C^* -algebras A without tracial states and improve the previous result of T. Fack. Note that if the unit of A is properly infinite, then A has no tracial states. The converse is known to be false, at least when A is non-simple (see [4] for further details). C^* -algebras without tracial states have several nice characterizations, such as [3]. This paper contains also another simple proof of the latter result of [3, Lemma 1].

2. THE RESULT

Given $a, b \in A$, their commutator is $[a, b] = ab - ba$. A self-adjoint commutator is just a commutator of the form $[a^*, a] = a^*a - aa^*$.

Theorem 1. *Let A be a unital C^* -algebra. Then the following are equivalent:*

1. *A has no tracial states.*
2. *There exist an integer $n \geq 2$, and elements $b_1, b_2, \dots, b_n \in A$ such that*

$$\begin{aligned} \sum_i b_i^* b_i &= 1 \\ \left\| \sum_i b_i b_i^* \right\| &< 1 \end{aligned}$$

3. *There exist an integer $n \geq 2$ such that any element of A can be expressed as a sum of n commutators and any positive element can be expressed as a sum of at most n self-adjoint commutators.*

Remark 2. The equivalence of 1 and 2 is just [3, Lemma 1]. As mentioned, we give here a new simple proof.

Remark 3. The integer n appearing in both 2 and 3 above represents the same number. Thus, if the unit of A is properly infinite, there exists two isometries $v_1, v_2 \in A$ with orthogonal ranges. Let $b_i = (1/\sqrt{2})v_i$ for $i = 1, 2$. Then $b_1^* b_1 + b_2^* b_2 = 1$ and $b_1 b_1^* + b_2 b_2^* = 1/2$, thus the property from 2 is achieved with $n = 2$.

Therefore positive elements are sums of 2 commutators and self-adjoint elements are sums of 4 commutators.

Proof. The implication $(3) \Rightarrow (1)$ is trivial.

$(1) \Rightarrow (2)$. Consider

$$R = \left\{ \sum_{i=1}^s (a_i^* a_i - a_i a_i^*) ; s \geq 1, a_i \in A \right\}$$

the set of **finite** sums of self-adjoint commutators of A . Note that $R \subset A_{sa}$ is a **real vector subspace** of A_{sa} . Put $\delta = \text{dist}(1, R)$.

We show that $\delta < 1$. Suppose the contrary, i.e. $\delta = 1$. This is equivalent to

$$\|t + x\| \geq |t|, \quad \forall x \in R.$$

It follows that $\varphi(t + x) = t$ is a real bounded functional on $\mathbb{R}1 + R$ of norm 1. By the Hahn–Banach theorem it can be extended to a norm-1 functional on A_{sa} and furthermore to a bounded **complex** functional on A , denoted also by φ . Observe that φ is necessarily a tracial state on A , which contradicts our hypothesis.

Because $\delta < 1$, there exist some elements $a_1, a_2, \dots, a_m \in A$ such that $t_0 = \|1 - \sum_{i=1}^m (a_i^* a_i - a_i a_i^*)\| < 1$. In particular we have

$$(1) \quad \sum_{i=1}^m a_i a_i^* \leq -1 + t_0 + \sum a_i^* a_i.$$

Let $k = \|\sum_{i=1}^m a_i^* a_i\|$ and $a_{m+1} = (k - \sum_{i=1}^m a_i^* a_i)^{1/2}$. Then we have

$$\sum_{i=1}^{m+1} a_i^* a_i = k$$

but on the other hand, by (1) we have also

$$\sum_{i=1}^{m+1} a_i a_i^* \leq -1 + t_0 + k.$$

The required properties are now fulfilled with $n = m + 1$ and $b_i = (1/\sqrt{k})a_i$.

$(2) \Rightarrow (3)$. Suppose that b_1, b_2, \dots, b_n are as in (3). Define $\phi(a) = \sum b_i a b_i^*$. Then ϕ is a bounded positive map on A with norm $\|\Phi\| = \|\sum b_i b_i^*\| < 1$. It follows that $Id_A - \Phi$ is invertible in the Banach algebra $\mathcal{B}(A)$ of bounded maps on A . Let

$$\Psi = (Id_A - \Phi)^{-1}.$$

Note that $\Psi = \sum_{i=0}^{\infty} \Phi^i$, thus Ψ is positive too. By definition of Ψ , for any $a \in A$ we have

$$a = (Id_A - \Phi)(\Psi(a)) = \Psi(a) - \sum_{i=1}^n b_i \Psi(a) b_i^* = \sum_{i=1}^n [b_i^*, b_i \Psi(a)],$$

so a is a finite sum of almost n commutators. If moreover a is a positive element in A then

$$a = (Id_A - \Phi)(\Psi(a)) = \Psi(a) - \sum_{i=1}^n b_i \Psi(a) b_i^* = \sum_{i=1}^n [\Psi(a)^{1/2} b_i^*, b_i \Psi(a)^{1/2}]$$

so a is a finite sum of at most n self-adjoint commutators. \square

3. QUESTIONS

For a infinite C^* -algebra A (in the sense of this paper) let $n(A)$ be the least positive integer such that any element of A is a sum of almost $n(A)$ commutators. In all the example we know, we have $n(A) = 2$. We believe that it's unlikely to be always the case.

In [3] it was shown that, if A is an unital exact C^* -algebra, then there exist an integer m such that $\mathbb{M}_m(A)$ is properly infinite. It follows that $n(\mathbb{M}_m(A)) = 2$. Then a simple computation shows that $n(A) \leq 2m^2$. It would be interesting to answer the inverse problem, that is: assuming $n(A)$ is known, estimate the least positive integer such that $n(\mathbb{M}_n(A)) = 2$.

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